2001
PAPER 1
(Non-Calc.)

1) Find the equation of the straight line which is parallel to the line with equation $2x + 3y = 5$ and which passes through the point $(2, -1)$.

Sol^n

Parallel lines have the **same** gradient.

$$2x + 3y = 5 \longrightarrow 3y = -2x + 5$$
$$y = -\frac{2}{3}x + \frac{5}{3}$$

The parallel line will have gradient $-\frac{2}{3}$ and passes through the point **(2, –1)**.

$y - b = m(x - a)$

$y + 1 = -\frac{2}{3}(x - 2)$

(multiply by 3 both sides)

$3y + 3 = -2(x - 2)$

$3y + 3 = -2x + 4$

$3y = -2x + 4 - 3$

$3y = -2x + 1$

$\mathbf{3y + 2x = 1}$

2) For what value of k does the equation $x^2 - 5x + (k + 6) = 0$ have equal roots?

Sol^n

For equal real roots $b^2 - 4ac = 0$

$a = 1$

$b = -5$

$c = (k + 6)$

$(-5)^2 - (4)(1)(k + 6) = 0$

$25 - 4(k + 6) = 0$

$25 - 4k + 24 = 0$

$1 - 4k = 0$

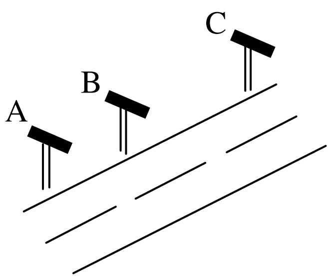
$1 = 4k$

$4k = 1$

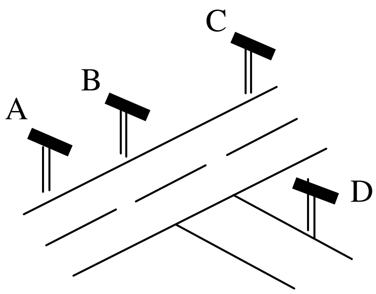
$\mathbf{k = \frac{1}{4}}$

3) (a) Roadmakers look along the tops of a set of T-rods to ensure that straight sections of road are being created. Relative to suitable axes he top left corners of the T-rods are the points A(−6, −10, −2), B(−2, −1, 1) and C(6, 11, 5).

Determine whether or not the section of road ABC has been built in a straight line.



(b) A further T-rod is placed such that D has co-ordinates (1, −4, 4). Show that DB is perpendicular to AB.



<u>Sol^n</u>

(a)

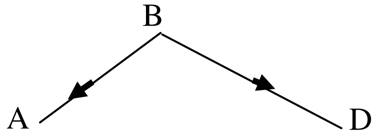
$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -8 \\ -10 \\ -2 \end{bmatrix} = \begin{bmatrix} -2+8 \\ -1+10 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\overrightarrow{BC} = \mathbf{c} - \mathbf{b} = \begin{bmatrix} 6 \\ 11 \\ 5 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6+2 \\ 11+1 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\mathbf{\tfrac{4}{3}\overrightarrow{AB} = \overrightarrow{BC}}$$

$\overrightarrow{AB}$ and $\overrightarrow{BC}$ are **parallel**, in **same direction** and have **common point B**.
A, B and C are collinear.
The section of road ABC has been built in a straight line.

(b) Required to show that DB is perpendicular to AB.



$$\cos \angle ABD = \frac{\overrightarrow{BA} \cdot \overrightarrow{BD}}{|BA|\,|BD|}$$

$$\overrightarrow{BA} = a - b = \begin{bmatrix} -8 \\ -10 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 + 2 \\ -10 + 1 \\ -2 - 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -9 \\ -3 \end{bmatrix}$$

$$\overrightarrow{BD} = d - b = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 + 2 \\ -4 + 1 \\ 4 - 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

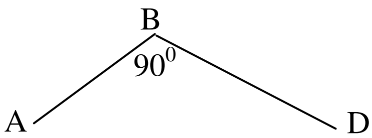
$$\begin{aligned} \overrightarrow{BA} \cdot \overrightarrow{BD} &= (-6 \times 3) + (-9 \times -3) + (-3 \times 3) \\ &= (-18) + (27) + (-9) \\ &= (-27) + (27) \\ &= \underline{\underline{0}} \end{aligned}$$

$$\cos \angle ABD = \frac{BA \cdot BD}{|BA|\,|BD|} = \frac{0}{|BA|\,|BD|} = 0$$

$\cos \angle ABD = 0$

$\cos 90^0 = 0$

$\angle ABD = 90^0$



DB is perpendicular to AB.

Additional

$$\overrightarrow{BA} = \begin{bmatrix} -6 \\ -9 \\ -3 \end{bmatrix} \qquad \overrightarrow{BD} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

$$BA = \sqrt{(-6)^2 + (-9)^2 + (-3)^2}$$
$$= \sqrt{36 + 81 + 9}$$
$$= \sqrt{126}$$
$$= \sqrt{9}\sqrt{14}$$
$$= \underline{\underline{3\sqrt{14}}}$$

$$BA = \sqrt{(3)^2 + (-3)^2 + (3)^2}$$
$$= \sqrt{9 + 9 + 9}$$
$$= \sqrt{27}$$
$$= \sqrt{9}\sqrt{3}$$
$$= \underline{\underline{3\sqrt{3}}}$$

4) Given $f(x) = x^2 + 2x - 8$, express $f(x)$ in the form $(x + a)^2 - b$.

<u>Solⁿ</u>

$f(x) = x^2 + 2x - 8$

Divide 2 by 2 = 1 ⟶ $f(x) = \mathbf{(x + 1)^2} - 9$

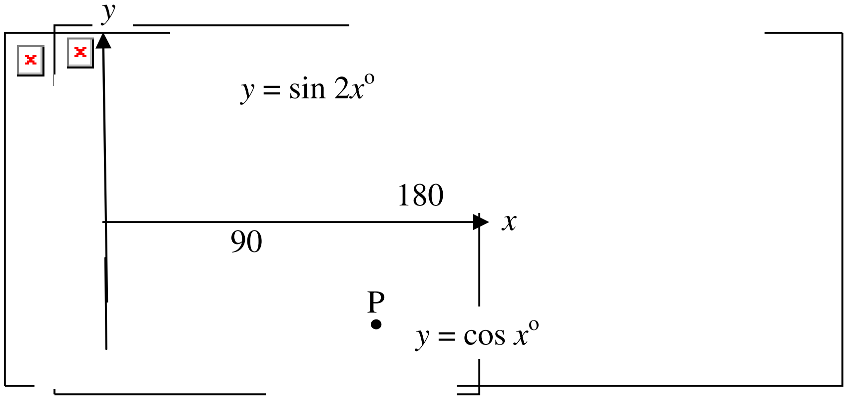
$= \mathbf{x^2 + 2x + 1} - 9$

$= x^2 + 2x - 8$

$\mathbf{f(x) = (x + 1)^2 - 9}$

5) (a) Solve the equation $\sin 2x^o - \cos x^o = 0$ in the interval $0 \le x \le 180$.

(b) The diagram shows parts of two trigonometric graphs, $y = \sin 2x^o$ and $y = \cos x^o$.
Use your solutions in (a) to write down the co-ordinates of the point P.



<u>Sol</u>

(a) $\sin 2x^o - \cos x^o = 0$

$2 \sin x^o \cos x^o - \cos x^o = 0$

$\cos x^o (2 \sin x^o - 1) = 0$

$\cos x^o = 0$ or $2 \sin x^o - 1 = 0$

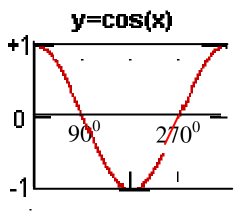
$\sin 2x^o$	$=$	$\sin(x + x)^o$
	$=$	$\sin x^o \cos x^o + \cos x^o \sin x^o$
	$=$	$2 \sin x^o \cos x^o$

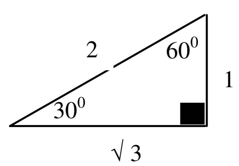
$\underline{\mathbf{x^o = 90^0}}$, $\underline{\mathbf{x^o = 270^0}}$ or $2\sin x^o = 1$

$\sin x^o = \frac{1}{2}$

$\underline{\mathbf{x^o = 30^0}}$, $x^o = (180 - 30)^0$

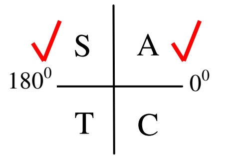
$\underline{\mathbf{x^o = 150^0}}$





$\mathbf{\sin 30^0} = \frac{1}{2}$ $\sin (180 - 30)^0 = \frac{1}{2}$

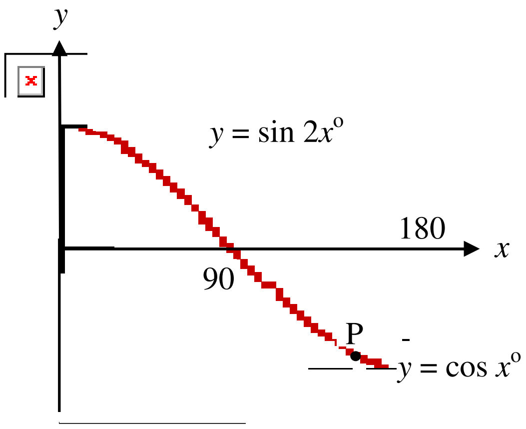
$\mathbf{\sin 150^0} = \frac{1}{2}$



sine is **positive** in the 1st and 2nd quadrants

<u>$0 \le x \le 180$</u> 3 Solutions $\underline{\underline{\mathbf{x^o = 30^0, 90^0, 150^0}}}$

(b)



P is a point of intersection of $y = \sin 2x^o$ and $y = \cos x^o$

$\sin 2x^o = \cos x^o$

$\sin 2x^o - \cos x^o = 0$

From (a) $\quad$ $\mathbf{x^o = 30^0, 90^0, 150^0}$

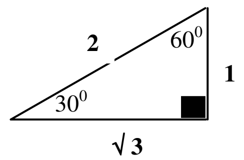
From the above diagram, P has x co-ordinate **150^0**

Using $\mathbf{y = \cos x^o}$

$\mathbf{y = \cos 150^0}$

$\mathbf{y = \dfrac{-\sqrt{3}}{2}}$

P is the point $\left(150^0, \dfrac{-\sqrt{3}}{2}\right)$

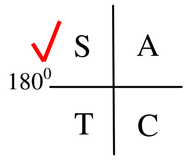


$\cos 30^0 = \dfrac{\sqrt{3}}{2}$

$\mathbf{\cos 150^0}$

$\mathbf{= \cos (180 - 30)^0}$

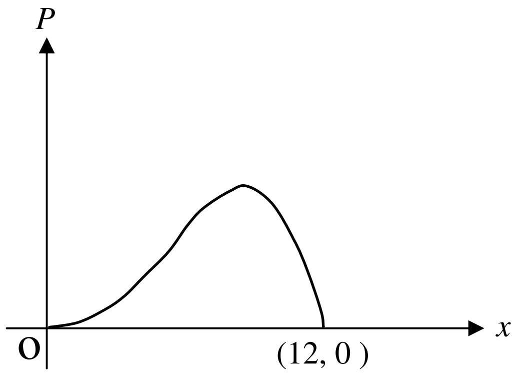
$\mathbf{= -\dfrac{\sqrt{3}}{2}}$



cosine is **negative** in the 2nd quadrant

6) A company spends x thousand pounds a year on advertising and this results in a profit of P thousand pounds. A mathematical model, illustrated in the diagram, suggests that P and x are related by $P = 12x^3 - x^4$ for $0 \le x \le 12$.

Find the value of x which gives the maximum profit.



Sol^n

$$P'(x) = 36x^2 - 4x^3$$
$$= 4x^2(9 - x)$$

Set $P'(x) = 0$

$4x^2(9 - x) = 0$

$4x^2 = 0$ or $(9 - x) = 0$

$\underline{\boldsymbol{x = 0}}$ or $9 = x$

$\underline{\boldsymbol{x = 9}}$

Nature

x	-1	0	1	9	10
$p'(x)$ $= 4x^2(9 - x)$	$4(9 + 1)$ $= 40$	0	$4(8)$ $= 32$	0	$400(-1)$ $= -400$
	/	—	/	—	\

The point (0, 0) is a point of inflection (increasing).

At $\underline{\boldsymbol{x = 9}}$ there is a maximum turning point.

x = 9 gives the maximum profit.

7) Functions $f(x) = \sin x$, $g(x) = \cos x$ and $h(x) = x + \frac{\pi}{4}$ are defined on a suitable set of real numbers.

(a) Find expressions for:
(i) $f(h(x))$;
(ii) $g(h(x))$.

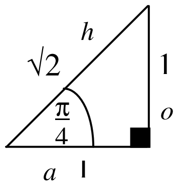
(b) (i) Show that $f(h(x)) = \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$.
(ii) Find a similar expression for $g(h(x))$ and hence solve the equation $f(h(x)) - g(h(x)) = 1$ for $0 \le x \le 2\pi$.

Sol^n

(a)(i) $f(h(x)) = \sin\left(x + \frac{\pi}{4}\right)$ (ii) $g(h(x)) = \cos\left(x + \frac{\pi}{4}\right)$

(b)(i) $f(h(x)) = \sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}$ (ii) $g(h(x)) = \cos x \cos\frac{\pi}{4} - \sin x \sin\frac{\pi}{4}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$



$f(h(x)) = \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$ $g(h(x)) = \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x$

Solve $\quad f(h(x)) - g(h(x)) = 1$ for $0 \le x \le 2\pi$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x - \left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right) = 1$$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x = 1$$

$$\frac{2}{\sqrt{2}}\sin x = 1$$

(multiply by √2 both sides)

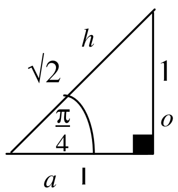
$$2\sin x = \sqrt{2}$$

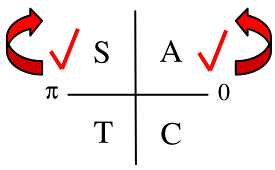
(divide by 2 both sides)

$$\sin x = \frac{\sqrt{2}}{2}$$

$$\sin x = \frac{\sqrt{2}}{\sqrt{2}\sqrt{2}}$$

$$\sin x = \frac{1}{\sqrt{2}}$$





sin is positive in both the 1st and 2nd quadrants

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin\left(\pi - \frac{\pi}{4}\right) = \sin\left(\frac{4\pi}{4} - \frac{\pi}{4}\right) = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

$$x = \frac{\pi}{4} \quad \text{or} \quad x = \frac{3\pi}{4}$$

8) Find x if $4 \log_x 6 - 2 \log_x 4 = 1$

Sol^n

$4 \log_x 6 - 2 \log_x 4 = 1$

$\log_x 6^4 - \log_x 4^2 = 1$

$\log_x \left(\frac{6^4}{4^2} \right) = 1$

$\log_x \frac{(6^2)(6^2)}{(4)(4)} = 1$

$\log_x \frac{(36)(36)}{(4)(4)} = 1$

$\log_x (9)(9) = 1$

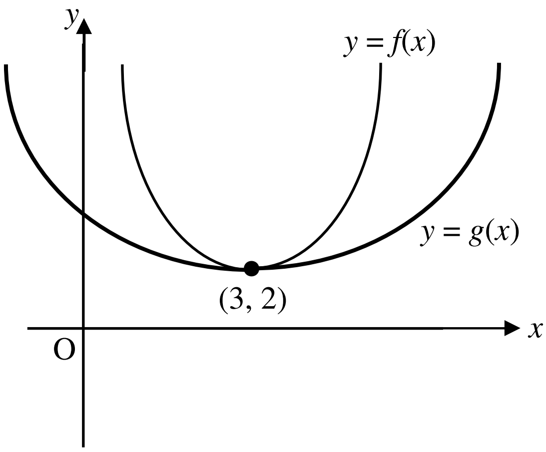
$\log_x 81 = 1$

$x^1 = 81$

$81^1 = 81$

$x = 81$

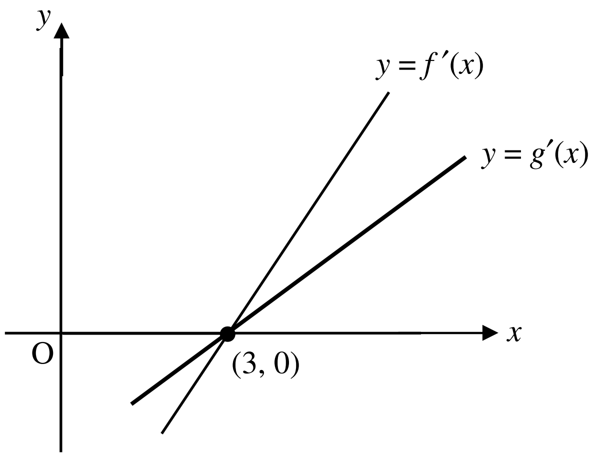
9) The diagram shows the graphs of two quadratic functions $y = f(x)$ and $y = g(x)$. Both graphs have a minimum turning point at (3, 2).
Sketch the graph of $y = f'(x)$ and on the same diagram sketch the graph of $y = g'(x)$.



<u>Solⁿ</u>

The graph of $y = f(x)$ has negative gradient as it approaches $x = 3$ from the left.
At $x = 3$ the gradient of $y = f(x)$ is 0.
The graph of $y = f(x)$ has positive gradient after $x = 3$.

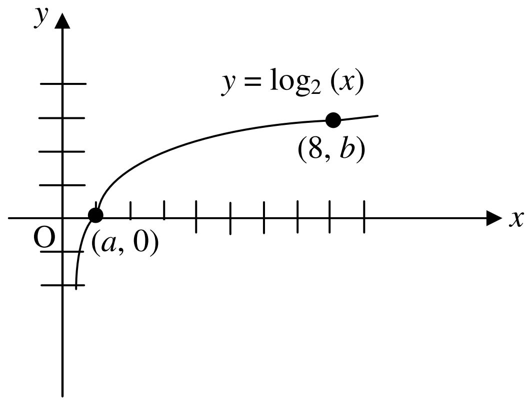
The same is true for $y = g(x)$.
However $y = g(x)$ has shallower negative gradient as it approaches $x = 3$ from the left and shallower positive gradient after $x = 3$.
At $x = 3$ the gradient of $y = g(x)$ is also 0.



10) The diagram shows a sketch of part of the graph of $y = \log_2(x)$.

(a) State the values of a and b.

(b) Sketch the graph of $y = \log_2(x + 1) - 3$



Sol^n

(a) $y = \log_2(x)$

$(a, 0)$

$y = 0 \quad x = a$

$0 = \log_2(a)$

$2^0 = a$

$\underline{\underline{a = 1}}$

$(a, 0)$ is the point $(1, 0)$

$y = \log_2(x)$

$(8, b)$

$y = b \quad x = 8$

$b = \log_2(8)$

$2^b = 8$

$\ln 2^b = \ln 8$

$b \ln 2 = \ln 8$

$$b = \frac{\ln 8}{\ln 2}$$

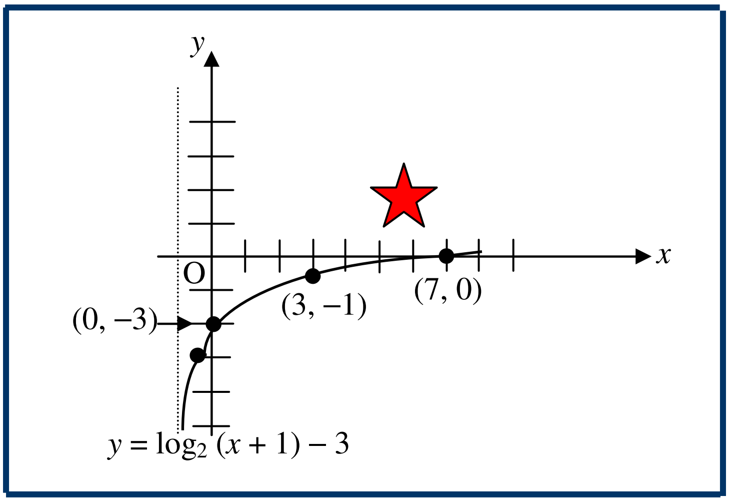
$\underline{\underline{b = 3}}$

$(8, b)$ is the point $(8, 3)$

(b) $y = \log_2 (x + 1) - 3$

is the graph of $y = \log_2 (x)$ moved **1 unit left** and **3 units down**.

$y = \log_2 (x)$	$y = \log_2 (x + 1)$ [subtract 1 from the x coordinates of $y = \log_2 (x)$]	$y = \log_2 (x + 1) - 3$ [subtract 3 from the y coordinates of $y = \log_2 (x + 1)$]
(1/4, –2)	(–3/4, –2)	(–3/4, –5)
(1/2, –1)	(–1/2, –1)	(–1/2, –4)
(1, 0)	(0, 0)	(0, –3)
(2, 1)	(1, 1)	(1, –2)
(4, 2)	(3, 2)	(3, –1)
(8, 3)	(7, 3)	(7, 0)



11) Circle P has equation $x^2 + y^2 - 8x - 10y + 9 = 0$. Circle Q has centre $(-2, -1)$ and radius $2\sqrt{2}$.

(a) (i) Show that the radius of circle P is $4\sqrt{2}$.
(ii) Hence show that circles P and Q touch.

(b) Find the equation of the tangent to circle Q at the point $(-4, 1)$.

(c) The tangent in (b) intersects circle P in two points. Find the x coordinates of the points of intersection, expressing your answers in the form $a \pm b\sqrt{3}$.

<u>Sol^n</u>

(a)(i) Equ^n of Circle P $\quad x^2 + y^2 - 8x - 10y + 9 = 0$

Equ^n of a circle $\quad x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -4$
$f = -5$
$c = 9$
$\longrightarrow$

$C_P = (-g, -f) = (4, 5)$

$r_P = \sqrt{g^2 + f^2 - c}$
$= \sqrt{(-4)^2 + (-5)^2 - 9}$
$= \sqrt{16 + 25 - 9}$
$= \sqrt{32}$
$= \sqrt{16}\sqrt{2}$

$r_P = 4\sqrt{2}$

(ii) Circle P has centre $(4, 5)$ $\longrightarrow$ $C_P\ (4, 5)$
Circle Q has centre $(-2, -1)$ $\longrightarrow$ $C_Q\ (-2, -1)$

If circles P and Q touch then the distance between the centres of the two circles will be the sum of the two radii

$|C_PC_Q| = r_P + r_Q = 4\sqrt{2} + 2\sqrt{2} = 6\sqrt{2}$

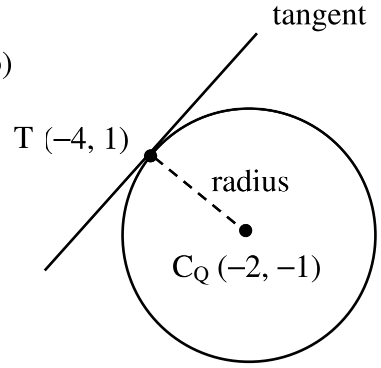
$|C_PC_Q| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ $\quad C_P\ (4, 5)$, $C_Q\ (-2, -1)$

$= \sqrt{(-2 - 4)^2 + (-1 - 5)^2}$
$= \sqrt{(-6)^2 + (-6)^2}$
$= \sqrt{36 + 36}$
$= \sqrt{72}$
$= \sqrt{36}\sqrt{2}$

$|C_PC_Q| = 6\sqrt{2}$

$|C_PC_Q| = r_P + r_Q = 6\sqrt{2}$ $\quad \therefore$ circles P and Q touch

(b)



$m_{rad} \times m_{tan} = -1$

$C_Q(-2, -1)$

$T(-4, 1)$

$$m_{rad} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{1 - (-1)}{(-4) - (-2)}$$

$$= \frac{1 + 1}{(-4) + 2}$$

$$= \frac{2}{(-2)}$$

$\underline{m_{rad} = -1}$

$\therefore \underline{m_{tan} = 1}$

Equation of tangent $\quad T(-4, 1)$, $m_{tan} = 1$ $\;\}\; \longrightarrow$

$y - b = m(x - a)$

$y - 1 = 1(x + 4)$

$y - 1 = x + 4$

(add 1 both sides)

$\underline{\underline{y = x + 5}}$

(c) Required to find the x-coordinates of the points of intersection of

$$x^2 + y^2 - 8x - 10y + 9 = 0$$

and $y = x + 5$

Substitute the line into the circle

$$x^2 + (x + 5)^2 - 8x - 10(x + 5) + 9 = 0$$

$$x^2 + x^2 + 10x + 25 - 8x - 10x - 50 + 9 = 0$$

$$2x^2 - 8x - 16 = 0$$

$$2(x^2 - 4x - 8) = 0$$

$$x^2 - 4x - 8 = 0$$

Quadratic Formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{4 \pm \sqrt{16 - (4)(1)(-8)}}{(2)(1)}$$

$$= \frac{4 \pm \sqrt{16 + 32}}{2}$$

$$= \frac{4 \pm \sqrt{48}}{2}$$

$$= \frac{4 \pm (\sqrt{16})(\sqrt{3})}{2}$$

$$= \frac{4 \pm 4\sqrt{3}}{2}$$

$$= 2 \pm 2\sqrt{3}$$

$x = 2 + 2\sqrt{3}$ or $x = 2 - 2\sqrt{3}$