2001
PAPER 2
(CALCULATOR)

1) (a) Given that $x + 2$ is a factor of $2x^3 + x^2 + kx + 2$, find the value of k.

(b) Hence solve the equation $2x^3 + x^2 + kx + 2 = 0$ when k takes this value.

Sol^n

(a) $x + 2$ is a factor

$x = -2$ is a root

$$\begin{array}{r|rrrr} -2 & 2 & 1 & k & 2 \\ & & -4 & 6 & -2k-12 \\ \hline & 2 & -3 & k+6 & \underline{\underline{0}} \\ & & & (-5+6) & \\ & & & 1 & \end{array}$$

$-2k - 12 + 2 = 0$

$-2k - 10 = 0$

$-2k = 10$

$\underline{\underline{k = -5}}$

(b) $2x^3 + x^2 + kx + 2 = 0$

$\boxed{k = -5}$

$2x^3 + x^2 - 5x + 2 = 0$

$(x + 2)(2x^2 - 3x + 1) = 0$

$(x + 2)(2x - 1)(x - 1) = 0$

$x + 2 = 0, \quad 2x - 1 = 0 \quad \text{or} \quad x - 1 = 0$

$\underline{\underline{x = -2}} \quad 2x = 1 \quad \text{or} \quad \underline{\underline{x = 1}}$

$\underline{\underline{x = \frac{1}{2}}}$

2) A curve has equation $y = x - \frac{16}{\sqrt{x}}$, $x > 0$

Find the equation of the tangent at the point where $x = 4$.

Sol^{n}

$\underline{x = 4}$

$y = 4 - \frac{16}{\sqrt{4}}$

$y = 4 - \frac{16}{2}$

$y = 4 - 8$

$\underline{y = -4}$

The point (4, − 4) lies on the curve.

$y = x - 16\,x^{-1/2}$

$\frac{dy}{dx} = 1 + 8\,x^{-3/2}$

$\frac{dy}{dx} = 1 + \frac{8}{x^{3/2}}$

$\frac{dy}{dx} = 1 + \frac{8}{(\sqrt{x})^3}$

$\underline{x = 4}$

$\frac{dy}{dx} = 1 + \frac{8}{(\sqrt{4})^3}$

$\frac{dy}{dx} = 1 + \frac{8}{(2)^3}$

$\frac{dy}{dx} = 1 + \frac{8}{8}$

$\frac{dy}{dx} = 1 + 1$

$\frac{dy}{dx} = 2$

The gradient of the tangent at the point $x = 4$ is 2

Point (4, – 4)
Gradient of the tangent = 2 → $y - b = m(x - a)$

$y + 4 = 2(x - 4)$

$y + 4 = 2x - 8$

$y = 2x - 8 - 4$

$\underline{y = 2x - 12}$

The equation of the tangent at the point where $x = 4$ is $\underline{\underline{y = 2x - 12}}$

3) On the first day of March, a bank loans a man £2500 at a fixed rate of interest of 1.5% per month. This interest is added on the last day of each month and is calculated on the amount due on the first day of the month. He agrees to make repayments on the first day of each subsequent month. Each repayment is £300 except for the smaller final amount which will pay off the loan.

(a) The amount that he owes at the start of each month is taken to be the amount still owing just after the monthly repayment has been made.

Let u_n and u_{n+1} represent the amounts that he owes at the starts of two successive months. Write down a recurrence relation involving u_{n+1} and u_n.

(b) Find the date and the amount of the final payment.

<u>Soln</u>

(a) $u_{n+1} = 1.015\, u_n - 300 \qquad u_0 = 2500$

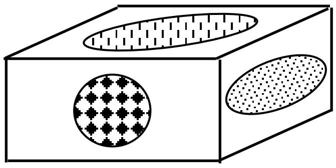
The amount owing at the start of the next month, is the amount from the start of the previous month with 1.5% interest added on and minus the repayment of £300.

(b)

	Amount Owing	Date
$u_0 \qquad\qquad =$	2500	1st March
$u_1 = (1.015)(2500) - 300 =$	2237.50	1st April
$u_2 = (1.015)(2237.50) - 300 =$	1971.06	1st May
$u_3 = (1.015)(1971.06) - 300 =$	1700.63	1st June
$u_4 = (1.015)(1700.63) - 300 =$	1426.14	1st July
$u_5 = (1.015)(1426.14) - 300 =$	1147.53	1st Aug.
$u_6 = (1.015)(1147.53) - 300 =$	864.74	1st Sept.
$u_7 = (1.015)(864.74) - 300 =$	577.71	1st Oct.
$u_8 = (1.015)(577.71) - 300 =$	286.38	1st Nov.
$u_8 = (1.015)(286.38) =$	290.68	1st Dec.

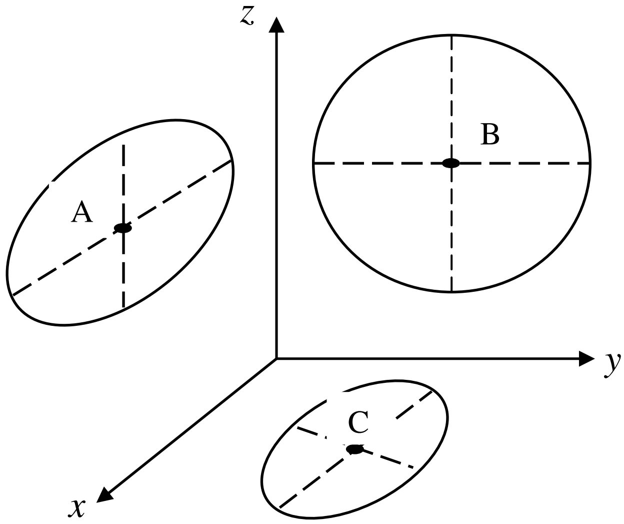
The date of the final payment is the <u>1st of December</u>
and
the amount is <u>£290.68</u>

4) A box in the shape of a cuboid is designed with **circles** of different sizes on each face.

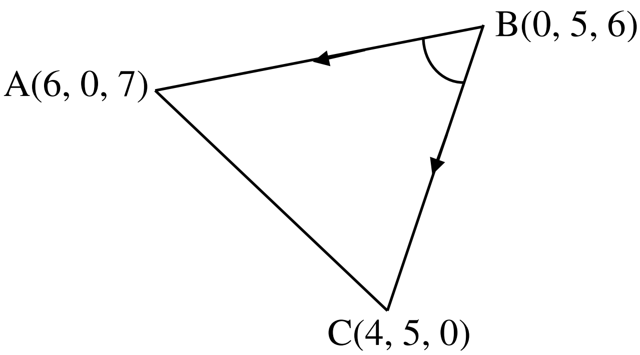


The diagram shows three of the circles, where the origin represents one of the corners of the cuboid.
The centres of the circles are A(6, 0, 7), B(0, 5, 6) and C(4, 5, 0).

Find the size of the acute angle ABC.



Sol^n



$$\cos\angle ABC = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|}$$

$$= \frac{18}{\sqrt{62}\sqrt{52}}$$

$$= 0.31701\ldots..$$

$$\angle ABC = 71.5^0 \text{ (1d.p.)}$$

$\overrightarrow{BA} = a - b$

$= \begin{pmatrix} 6 \\ 0 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ -5 \\ 1 \end{pmatrix}$

$\overrightarrow{BC} = c - b$

$= \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix}$

$\overrightarrow{BA} \cdot \overrightarrow{BC}$
$= (6 \times 4) + (-5 \times 0) + (1 \times -6)$
$= 24 + 0 + (-6)$
$= 24 - 6$
$= 18$

$|\overrightarrow{BA}| = \sqrt{6^2 + (-5)^2 + 1^2} = \sqrt{62}$

$|\overrightarrow{BC}| = \sqrt{4^2 + 0^2 + (-6)^2} = \sqrt{52}$

5) Express $8\cos x^\circ - 6\sin x^\circ$ in the form $k\cos(x + a)^\circ$ where $k > 0$ and $0 < a < 360$.

<u>Sol</u>

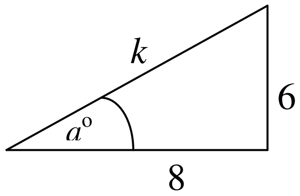
$$k\cos(x+a)^\circ = k\cos x^\circ\cos a^\circ - k\sin x^\circ\sin a^\circ$$

$$8\cos x^\circ - 6\sin x^\circ$$

$k\cos a^\circ = 8$ $\quad$ $k\sin a^\circ = 6$

$\cos a^\circ = \dfrac{8}{k}$ $\quad$ $\sin a^\circ = \dfrac{6}{k}$

$\cos a^\circ = \dfrac{8}{10}$ $\quad$ $\sin a^\circ = \dfrac{6}{10}$



$k = \sqrt{8^2 + 6^2}$
$k = \sqrt{64 + 36}$
$k = \sqrt{100}$
$\underline{\underline{k = 10}}$

cos and sin are both positive
$\therefore$ a° lies in the 1st quadrant

$\cos a^\circ = \dfrac{8}{10}$

$\cos a^\circ = 0.8$

$\underline{\underline{a^\circ = 36.9^0}}$ (1 d.p.)

$$8\cos x^\circ - 6\sin x^\circ = k\cos(x+a)^\circ$$

$\underline{k = 10}$ $\quad$ $\underline{a^\circ = 36.9^0}$

$$8\cos x^\circ - 6\sin x^\circ \;\underline{\underline{= 10\cos(x + 36.9)^0}}$$

6) Find $\int \frac{(x^2-2)(x^2+2}{x^2}\ dx\ ,\ \ x \neq 0$

<u>Sol^n</u>

$$\int \frac{(x^2-2)(x^2+2)}{x^2}\ dx$$

$$= \int \frac{x^4-4}{x^2}\ dx$$

$$= \int \frac{x^4}{x^2} - \frac{4}{x^2}\ dx$$

$$= \int x^2 - 4x^{-2}\ dx$$

$$= \frac{x^3}{3} - \frac{4x^{-1}}{(-1)} + c$$

$$= \frac{x^3}{3} + \frac{4}{x} + c$$

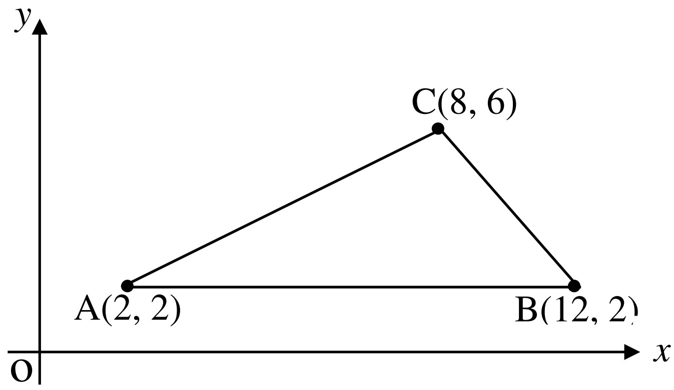
7) Triangle ABC has vertices A(2, 2), B(12, 2) and C(8, 6).

(a) Write down the equation of l_1, the perpendicular bisector of AB.

(b) Find the equation of l_2, the the perpendicular bisector of AC.

(c) Find the point of intersection of lines l_1 and l_2.

(d) Hence find the equation of the circle passing through A, B and C.



Sol^n

(a) Find the mid-point of AB.

For the mid-point of AB use the formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

A(2, 2)

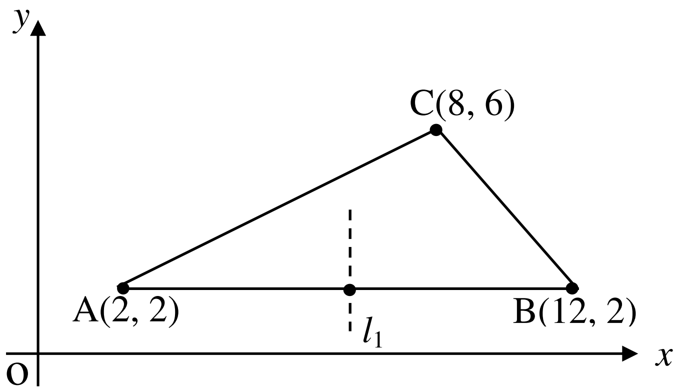
B(12, 2) $\left(\frac{2 + 12}{2}, \frac{2 + 2}{2} \right)$

$\left(\frac{14}{2}, \frac{4}{2} \right)$

The mid-point of AB is (7, 2)

The perpendicular bisector of AB, l_1 , passes through the point (7, 2) and is parallel to the y axis.

The equation of l_1 is $x = 7$



(b) Find the mid-point of AC.

For the mid-point of AC use the formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

A(2, 2)

C(8, 6) $\left(\frac{2+8}{2}, \frac{2+6}{2} \right)$

$\left(\frac{10}{2}, \frac{8}{2} \right)$

The mid-point of AC is (5, 4)

Find the gradient of AC ($m_{AC} \times m_{l2} = -1$)

A(2, 2)
C(8, 6)

$$m_{AC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{6-2}{8-2}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

The gradient of the perpendicular bisector of AC, l_2, is $\frac{-3}{2}$

l_2 passes through the point (5, 4)

$y - b = m(x - a)$
$y - 4 = \frac{-3}{2}(x - 5)$
($\times$ 2 both sides)
$2y - 8 = -3(x - 5)$
$2y - 8 = -3x + 15$
$2y = -3x + 15 + 8$
$2y = -3x + 23$

The equation of l_2 is $2y = -3x + 23$

(c) Required to find the point of intersection of l_1 and l_2

i.e. the point of intersection of $x = 7$
and $2y = -3x + 23$

$2y = -3x + 23$

$\underline{x = 7}$

$2y = -3(7) + 23$

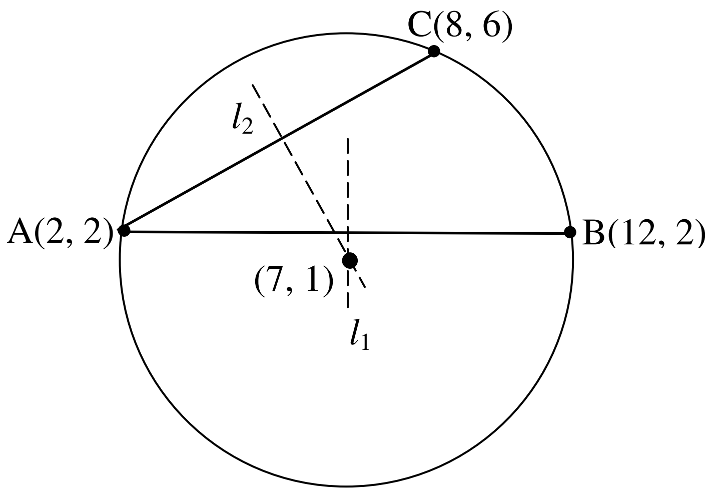
$2y = -21 + 23$

$2y = 2$

$\underline{y = 1}$

The point of intersection of l_1 and l_2 is (7, 1)

(d) A circle passes through A, B and C



AB and AC are chords of the circle

The perpendicular bisectors of AB and AC, l_1 and l_2 , intersect at the centre of the circle

The centre of the circle is the point (7, 1)

A, B and C lie on the circle

The length of the radius is the distance from the centre to A, B or C

Choose to find the distance from A(2, 2) to the centre (7, 1)

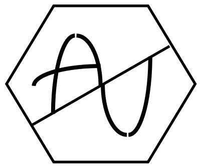
$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(7-2)^2 + (1-2)^2}$$
$$= \sqrt{5^2 + (-1)^2}$$
$$= \sqrt{25 + 1}$$
$$= \sqrt{26}$$

The length of the radius of the circle is $\sqrt{26}$
The centre of the circle is (7, 1)

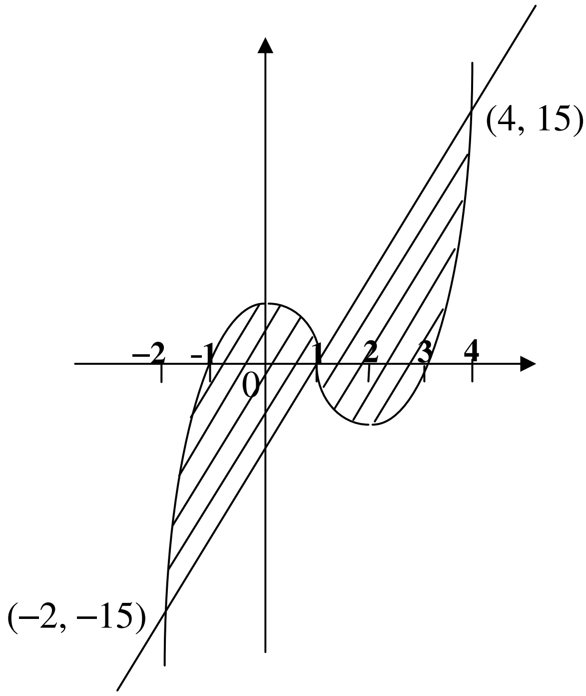
The equation of the circle is $(x - a)^2 + (y - b)^2 = r^2$

$$\underline{\underline{(x - 7)^2 + (y - 1)^2 = 26}}$$

8) A firm asked for a logo to be designed involving the letters A and U. Their initial sketch is shown in the hexagon.



A mathematical representation of the final logo is shown in the coordinate diagram.



The curve has equation $y = (x + 1)(x - 1)(x - 3)$ and the straight line has equation $y = 5x - 5$. The point (1, 0) is the centre of the half-turn symmetry.

Calculate the total shaded area.

<u>Sol^n</u>

The logo is symmetrical

The shaded area is $$2 \times \int_1^4 (5x - 5) - [(x + 1)(x - 1)(x - 3)]\, dx$$

$$= 2 \times \int_1^4 (5x - 5) - [(x^2 - 1)(x - 3)]\, dx$$

$$= 2 \times \int_1^4 5x - 5 - (x^3 - 3x^2 - x + 3)\ \ dx$$

$$= 2 \times \int_1^4 6x - x^3 + 3x^2 - 8\, dx$$

$$= 2 \times \left[\frac{6x^2}{2} - \frac{x^4}{4} + 3\frac{x^3}{3} - 8x \right]_1^4$$

$$= 2 \times \left[3x^2 - \frac{x^4}{4} + x^3 - 8x \right]_1^4$$

$$= 2 \times \left[\left(3(4)^2 - \frac{(4)^4}{4} + (4)^3 - 8(4)\right) - \left(3(1^2) - \frac{1^4}{4} + 1^3 - 8(1)\right) \right]$$

$$= 2 \times \left[\left(48 - \frac{256}{4} + 64 - 32\right) - \left(3 - \frac{1}{4} + 1 - 8\right) \right]$$

$$= 2 \times \left[48 - 64 + 64 - 32 - 3 + \frac{1}{4} - 1 + 8 \right]$$

$$= 2 \times \left[20\frac{1}{4} \right]$$

$$= 40\frac{1}{2}$$

<u>The shaded area is $40\frac{1}{2}$ uints^2</u>

9) Before a forest fire was brought under control, the spread of the fire was described by a law of the form $A = A_0 e^{kt}$ where A_0 is the area covered by the fire when it was first dected and A is the area covered by the fire t hours later.

If it takes one and a half hours for the area of the forest fire to double, find the value of the constant k.

<u>Sol^{n}</u>

$A = A_0 e^{kt}$	$A = 2A_0$ $t = 1.5$	The area covered, after 1.5 hours, is double the original area first detected

$2A_0 = A_0 e^{1.5k}$

(divide by A_0 both sides)

$2 = e^{1.5k}$

(take the ln i.e. $\log_e$ both sides)

$\ln 2 = \ln e^{1.5k}$

$\ln 2 = (1.5k)(\ln e)$ [$\ln e = \log_e e = 1$]

$\ln 2 = 1.5k$

$\dfrac{\ln 2}{1.5} = k$

$k = 0.462$ (3 decimal places)

10) A curve for which $\frac{dy}{dx} = 3\sin(2x)$ passes through the point $\left(\frac{5}{12}\pi, \sqrt{3}\right)$

Find y in terms of x.

Sol^n

$$y = \int 3\sin(2x)\, dx$$

$$y = -3\cos(2x) \cdot \frac{1}{2} + c \qquad\qquad y = \frac{-3\cos(2x)}{2} + c$$

The curve passes through the point $\left(\frac{5}{12}\pi, \sqrt{3}\right)$

i.e. $x = \frac{5}{12}\pi$ and $y = \sqrt{3}$

$$\sqrt{3} = -3\cos\left(2 \cdot \frac{5}{12}\pi\right) \cdot \frac{1}{2} + c$$

$$\sqrt{3} = \frac{-3}{2}\cos\frac{10}{12}\pi + c$$

$$\sqrt{3} = \frac{-3}{2}\cos\frac{5}{6}\pi + c \qquad\qquad \cos\frac{5}{6}\pi = -\frac{\sqrt{3}}{2}$$

$$\sqrt{3} = \frac{-3}{2} \cdot -\frac{\sqrt{3}}{2} + c$$

$$\sqrt{3} = \frac{3\sqrt{3}}{4} + c$$

($\times$ 4 both sides)

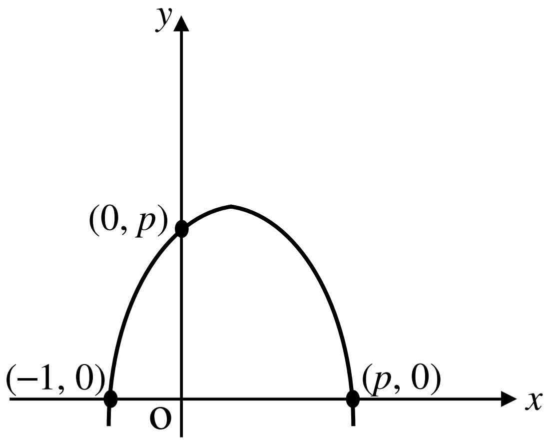
$$4\sqrt{3} = 3\sqrt{3} + 2c$$

$$2c = \sqrt{3}$$

$$c = \frac{\sqrt{3}}{4}$$

$$y = \frac{-3\cos(2x)}{2} + \frac{\sqrt{3}}{4}$$

11) The diagram shows a sketch of a parabola passing through $(-1, 0)$. $(0, p)$ and $(p, 0)$.



(a) Show that the equation of the parabola is $y = p + (p - 1)x - x^2$.

(b) For what values of p will the line $y = x + p$ be a tangent to this curve.

Sol^n

(a) $y = -(x + 1)(x - p)$

$y = -(x^2 - px + x - p)$

$y = -x^2 + px - x + p$

$y = p + px - x - x^2$

$y = p + (p - 1)x - x^2$ ★

(b) Curve $y = p + (p-1)x - x^2$
Line $y = x + p$

Set equal $p + (p-1)x - x^2 = x + p$

$p + px - x - x^2 = x + p$

$p + px - x - x^2 - x - p = 0$

$px - 2x - x^2 = 0$

$(p-2)x - x^2 = 0$

For equal real roots $b^2 - 4ac = 0$

$a = -1 \qquad b = p - 2 \qquad c = 0$

$b^2 - 4ac = (p-2)^2 - 4(-1)(0)$

$b^2 - 4ac = (p-2)^2$

Set $(p-2)^2 = 0$

$p - 2 = 0$

$p = 2$