MATHEMATICS HIGHER

Units 1, 2 and 3

2002
Paper I

Questions
&
Worked Solutions

1) The point P(2,3) lies on the circle $(x + 1)^2 + (y - 1)^2 = 13$. Find the equation of the tangent at P.

Sol^n

Obtain the centre of the circle

Use the formula $(x - a)^2 + (y - b)^2 = r^2$

$(x + 1)^2 + (y - 1)^2 = 13$

Centre of the circle is the point (a,b)

Centre of the circle is the point C (-1,1)

Obtain the gradient of the radius CP

Require the gradient of the radius CP.

The radius CP is perpendicular to the tangent at P.

C (-1,1)

P (2,3)

$$m_{CP} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{2 - (-1)} = \frac{2}{2+1} = \frac{2}{3}$$

Obtain the gradient of the tangent at P

$m_{CP} \times m_{\text{tangent at P}} = -1$

$\frac{2}{3} \times \frac{-3}{2} = -1$

Gradient of the tangent at P is $\frac{-3}{2}$

Obtain the equation of the tangent at P

P(2,3)

$m_{\text{tangent at P}} = \frac{-3}{2}$

$y - b = m(x - a)$

$y - 3 = \frac{-3}{2}(x - 2)$

(x by 2 both sides)

$2y - 6 = -3(x - 2)$

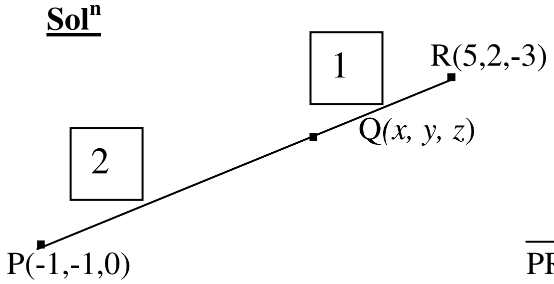
$2y - 6 = -3x + 6$

$2y = -3x + 6 + 6$

$2y = -3x + 12$

2) The point Q divides the line joining P(-1,-1,0) to R(5,2,-3) in the ratio 2:1. Find the coordinates of Q.

Sol^n



$\overrightarrow{PR} = \underline{r} - \underline{p}$

(3 parts) $= \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}$

$= 3 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

1 part is $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\overrightarrow{QR} = \underline{r} - \underline{q}$

(1 part) $\begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$5 - x = 2$
$2 - y = 1$
$-3 - z = -1$

$\longrightarrow$

$x = 3$
$y = 1$
$z = -2$

$\longrightarrow$ $Q(x, y, z) = \underline{\underline{Q(3,1,-2)}}$

<u>Check</u> $\overrightarrow{PQ} = \underline{q} - \underline{p}$

(2 parts) $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

3) Functions f and g are defined on a suitable domain by $f(x) = \sin(x^o)$ and $g(x) = 2x$.

a) Find expressions for:

(i) $f(g(x)$;
(ii) $g(f(x))$.

b) Solve $2f(g(x)) = g(f(x))$ for $0 \leq x \leq 360$.

Sol[n]

a)(i) $f(g(x)) = \sin 2x^o$

$f(x) = \sin(x^o)$
$g(x) = 2x$

a)(ii) $g(f(x)) = 2\sin x^o$

$g(x) = 2x$
$f(x) = \sin(x^o)$

b) $2f(g(x)) = g(f(x))$ $\quad 0 \leq x \leq 360$

$2\sin 2x^o = 2\sin x^o$
$\sin 2x^o = \sin x^o$
$\sin 2x^o - \sin x^o = 0$

$2\sin x^o\cos x^o - \sin x^o = 0$
$\sin x^o(2\cos x^o - 1) = 0$

$\sin x^o = 0$ or $2\cos x^o - 1 = 0$

$x^o = 0^o, 180^o, 360^o$

$\cos x^o = \frac{1}{2}$

$x^o = 60^o, 300^o$

$\sin(2x^o) = \sin(x + x)^o$
$= \sin x^o\cos x^o + \sin x^o\cos x^o$
$= 2\sin x^o\cos x^o$

$\cos 60^o = \frac{1}{2}$
$\cos(360 - 60)^o$
$= \cos 300^o$
$= \frac{1}{2}$

4) Find the coordinates of the point P on the curve $y = 2x^2 - 7x + 10$ where the tangent to the curve makes an angle of 45^0 with the positive direction of the x-axis.

Sol^n

The gradient of the tangent is $\tan 45^0$ i.e. $m_{tangent} = \tan 45^0 = 1$

To find the gradient of the tangent to the curve at any point on the curve differentiate. $\frac{dy}{dx} = 4x - 7$

$\frac{dy}{dx} = 4x - 7 = 1$

$4x - 7 = 1$

$4x = 8$

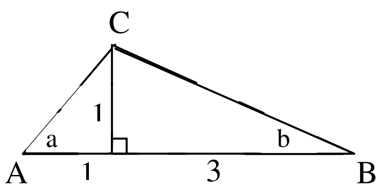
$x = 2$

$y = 2x^2 - 7x + 10$
$= 2(2^2) - 7(2) + 10$
$= 2(4) - 14 + 10$
$= 8 - 14 + 10$
$= 4$

$y = 4$

P is the point P(2,4)

5) In triangle ABC, show that the exact value of sin(a + b) is $\frac{2}{\sqrt{5}}$.



Sol^n

$\sin(a + b) = (\sin a)(\cos b) + (\cos a)(\sin b)$

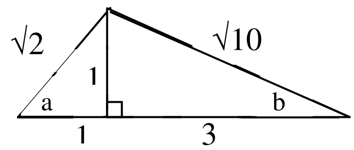
$= \frac{1}{\sqrt{2}} \cdot \frac{3}{\sqrt{10}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{10}}$

$= \frac{3}{\sqrt{20}} + \frac{1}{\sqrt{20}}$

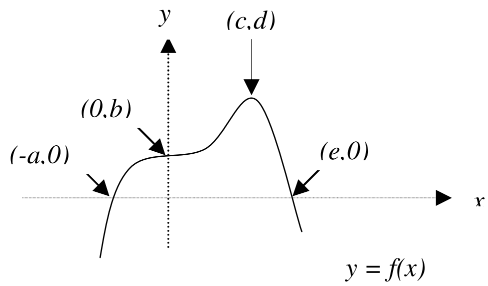
$= \frac{4}{\sqrt{20}}$

$= \frac{\sqrt{4}\sqrt{4}}{\sqrt{4}\sqrt{5}}$

$= \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}}$



6) The graph of a function f intersects the x-axis at $(-a,0)$ and $(e,0)$ as shown.
There is a point of inflection at $(0,b)$ and a maximum turning point at (c,d).
Sketch the graph of the derived function $f\,'$.



Sol^n

At $x = 0$ the gradient of $f(x) = 0$ → $x = 0$ $f\,'(x) = 0$
$x = c$ the gradient of $f(x) = 0$ → $x = c$ $f\,'(x) = 0$

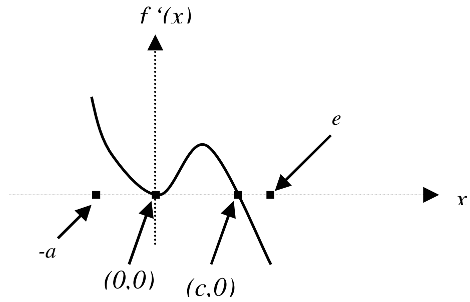
The points $(0,0)$ and $(c,0)$ lie on the graph of the derived function $f\,'$

From $-a$ to 0 the gradient of $f(x)$ is positive (i.e. $f\,'(x) > 0$)

From 0 to c the gradient of $f(x)$ is positive (i.e. $f\,'(x) > 0$)

From c to e the gradient of $f(x)$ is negative (i.e. $f\,'(x) < 0$)

Sketch of the graph of the derived function $f\,'$



7) (a) Express $f(x) = x^2 - 4x + 5$ in the form $f(x) = (x - a)^2 + b$.

(b) On the same diagram sketch:

(i) the graph of $y = f(x)$;

(ii) the graph of $y = 10 - f(x)$.

(c) Find the range of values of x for which $10 - f(x)$ is positive.

Sol^n

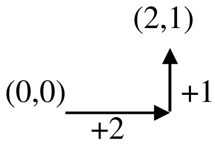
(a) $f(x) = x^2 - 4x + 5$

$f(x) = (x - 2)^2 + 1$

$(x - 2)^2 + 1 = x^2 - 4x + 4 + 1 = x^2 - 4x + 5$

(b) minimum turning point at (2,1)

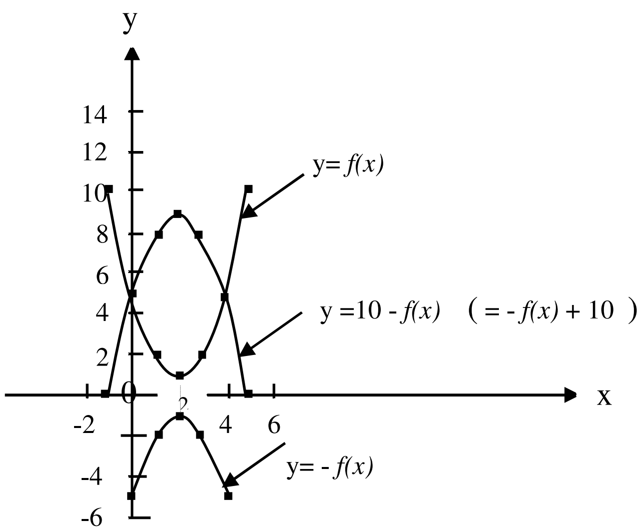
Graph of $g(x) = x^2$ has moved 2 places **right** and **up** 1 i.e.



Graph of $f(x) = x^2 - 4x + 5$

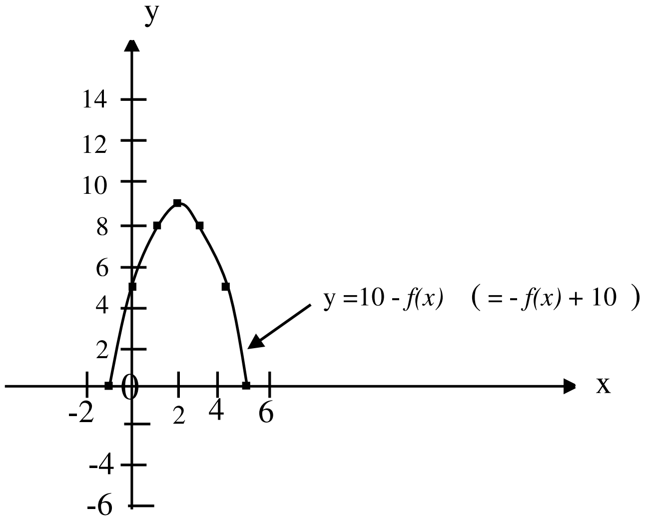
$f(x) = (x - 2)^2 + 1$

$g(x) = x^2$	$f(x) = (x - 2)^2 + 1$
(0,0)	(2,1)
(1,1)	(3,2)
(2,4)	(4,5)
(3,9)	(5,10)
(-1,1)	(1,2)
(-2,4)	(0,5)
(-3,9)	(-1,10)



$y = f(x) = (x - 2)^2 + 1$	$y = -f(x)$ (multiply y co-ordinate of $y = f(x)$ by –1 i.e. reflect graph of $y = f(x)$ in the x-axis)	$y = -f(x) + 10 = 10 - f(x)$ (add 10 to the y co-ordinate of $y = -f(x)$) i.e. move graph of $y = -f(x)$ up 10)
(2,1)	(2,-1)	(2,9)
(3,2)	(3,-2)	(3,8)
(4,5)	(4,-5)	(4,5)
(5,10)	(5,-10)	(5,0)
(1,2)	(1,-2)	(1,8)
(0,5)	(0,-5)	(0,5)
(-1,10)	(-1,-10)	(-1,0)

(c)



$y = -f(x) + 10 = 10 - f(x)$

(2,9)
(3,8)
(4,5)
(5,0)
(1,8)
(0,5)
(-1,0)

From the diagram opposite and the points listed above, the graph of $y = -f(x) + 10 = 10 - f(x)$ intercepts the x-axis at the points (-1,0) and (5,0).

$y = -f(x) + 10 = 10 - f(x)$ is positive (i.e. y >0) when $-1 < x < 5$ ★

Algebraically

$$\begin{aligned} y &= -f(x) + 10 \\ &= -(x^2 - 4x + 5) + 10 \\ &= -x^2 + 4x - 5 + 10 \\ &= -x^2 + 4x + 5 \\ &= -(x^2 - 4x - 5) \\ &= -(x - 5)(x + 1) \end{aligned}$$

Set $y = 0$

$$-(x - 5)(x + 1) = 0$$

$(x - 5) = 0$ or $(x + 1) = 0$

$x = 5$ or $x = -1$

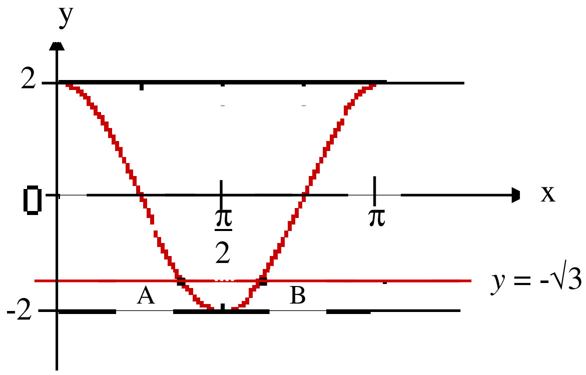
The points (-1,0) and (5,0) lie on the curve $y = -f(x) + 10 = 10 - f(x)$.

$y = -f(x) + 10 = 10 - f(x)$ has a maximum turning point, so is positive when $-1 < x < 5$ ★

8) The diagram shows the graph of a cosine function from 0 to π.

(a) State the equation of the graph.

(b) The line with equation $y = -\sqrt{3}$ intersects the graph at points A and B. Find the co-ordinates of B.



<u>Sol^n</u>

(a) $y = a \cos bx$

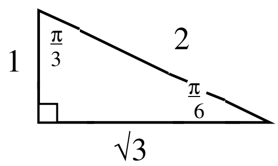
<u>y-axis</u> maximum value = 2, minimum value = -2 } $a = 2$ → $y = 2 \cos bx$

The above cosine graph has a period of π radians (i.e. repeats <u>twice</u> in a 2π interval) } $b = 2$ → $y = 2 \cos 2x$ ★

(b) A and B are the points of intersection of $y = 2 \cos 2x$ and $y = -\sqrt{3}$ ($0 \le x \le \pi$)

i.e. $2 \cos 2x = -\sqrt{3}$

$\cos 2x = \frac{-\sqrt{3}}{2}$



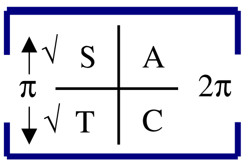
$2x = \frac{5\pi}{6}$ or $2x = \frac{7\pi}{6}$

(for both equations, divide by 2 both sides)

$x = \frac{5\pi}{12}$ or $x = \frac{7\pi}{12}$

$A\left(\frac{5\pi}{12}, -\sqrt{3}\right)$ $B\left(\frac{7\pi}{12}, -\sqrt{3}\right)$ ★

$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$



$\cos\left[\pi - \frac{\pi}{6}\right] = \frac{-\sqrt{3}}{2}$ or $\cos\left[\pi + \frac{\pi}{6}\right] = \frac{-\sqrt{3}}{2}$

$\cos \frac{5\pi}{6} = \frac{-\sqrt{3}}{2}$ or $\cos \frac{7\pi}{6} = \frac{-\sqrt{3}}{2}$

9) (a) Write sin(x) – cos(x) in the form k sin(x – a) stating the values of k and a where $k > 0$ and $0 \le a \le 2\pi$.

(b) Sketch the graph of $y = \sin(x) - \cos(x)$ for $0 \le x \le 2\pi$, showing clearly the graph's maximum and minimum values and where it cuts the x-axis and the y-axis.

<u>Sol^n</u>

(a)

$k \sin(x - a) = \sin x - \cos x$

$k \sin x \cos a - k \cos x \sin a = \sin x - \cos x$

$k \sin x \cos a - k \cos x \sin a = 1 . \sin x - 1 . \cos x$

Sine and Cosine are **<u>both positive</u>**

a is in the **<u>first quadrant</u>**

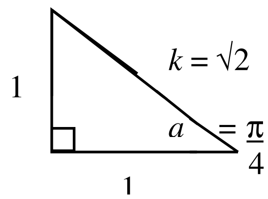
S	A √
T	C

$k \cos a = 1$ $\quad$ $k \sin a = 1$

$\cos a = \frac{1}{k}$ $\quad$ $\sin a = \frac{1}{k}$

$\cos a = \frac{1}{\sqrt{2}}$ $\quad$ $\sin a = \frac{1}{\sqrt{2}}$

$a = \frac{\pi}{4}$



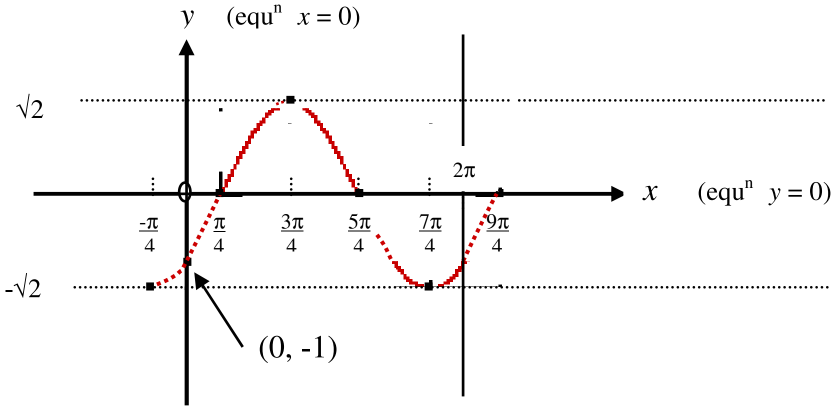
By Pythagoras:
$k = \sqrt{(1^2 + 1^2)}$
$= \sqrt{(1 + 1)}$
$= \sqrt{2}$

$k = \sqrt{2}$

$\sin x - \cos x = k \sin(x - a)$

$\sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$

(b) $y = \sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$



$y = \sqrt{2}\sin(x - \frac{\pi}{4})$

Graph of $y = \sin x$ has moved $\frac{\pi}{4}$ radians **right**

$\longrightarrow y = \sin(x - \frac{\pi}{4})$

Multiply the y co-cordinates of $y = \sin(x - \frac{\pi}{4})$

by $\sqrt{2}$ $\longrightarrow$ $y = \sqrt{2}\sin(x - \frac{\pi}{4})$

Maximum value on the y-axis is $\sqrt{2}$
Minimum value on the y-axis is $-\sqrt{2}$

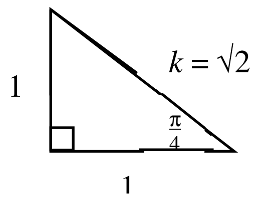
Maximum Turning Point	$(\frac{3\pi}{4}, \sqrt{2})$	Cuts x-axis at points	$(\frac{\pi}{4}, 0)$ and $(\frac{5\pi}{4}, 0)$
Minimum Turning Point	$(\frac{7\pi}{4}, -\sqrt{2})$	Cuts y-axis at point	$(0, -1)$

$y = \sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$ for $0 \le x \le 2\pi$ cuts the y-axis when $x = 0$

Set $x = 0$

$$y = \sqrt{2}\sin(x - \frac{\pi}{4})$$
$$y = \sqrt{2}\sin(0 - \frac{\pi}{4})$$
$$y = \sqrt{2}\sin(-\frac{\pi}{4})$$
$$y = \sqrt{2} \cdot \frac{-1}{\sqrt{2}}$$
$$y = -1$$

$\sin(-\frac{\pi}{4}) = -\sin\frac{\pi}{4}$
$= \frac{-1}{\sqrt{2}}$



$y = \sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$ cuts the y-axis at the point $(0, -1)$

Points on the graph of $y = \sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$ for $0 \le x \le 2\pi$

can be obtained as detailed below.

$y = \sin x$	Add $\frac{\pi}{4}$ to the x co-ords of $y = \sin x$ →	$y = \sin(x - \frac{\pi}{4})$	Multiply the y co-ords of $y = \sin(x - \frac{\pi}{4})$ by $\sqrt{2}$ →	$y = \sqrt{2}\sin(x - \frac{\pi}{4}) = \sin x - \cos x$
$(0, 0)$		$(\frac{\pi}{4}, 0)$		$(\frac{\pi}{4}, 0)$ ★
$(\frac{\pi}{2}, 1)$		$(\frac{3\pi}{4}, 1)$		$(\frac{3\pi}{4}, \sqrt{2})$
$(\pi, 0)$		$(\frac{5\pi}{4}, 0)$		$(\frac{5\pi}{4}, 0)$ ★
$(\frac{3\pi}{2}, -1)$		$(\frac{7\pi}{4}, -1)$		$(\frac{7\pi}{4}, -\sqrt{2})$
$(2\pi, 0)$		$(\frac{9\pi}{4}, 0)$		$(\frac{9\pi}{4}, 0)$

$y = \sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$ for $0 \le x \le 2\pi$ cuts the x-axis when $y = 0$

i.e. at points $(\frac{\pi}{4}, 0)$ and $(\frac{5\pi}{4}, 0)$

10) (a) Find the derivative of the function $f(x) = (8 - x^3)^{\frac{1}{2}}$, $x < 2$.

(b) Hence write down $\int \frac{x^2}{(8 - x^3)^{\frac{1}{2}}}\, dx$

Sol^n

(a) $f'(x) = \frac{1}{2}(8 - x^3)^{\frac{-1}{2}} \cdot -3x^2$

$= \frac{-3x^2}{2(8 - x^3)^{\frac{1}{2}}}$ ★

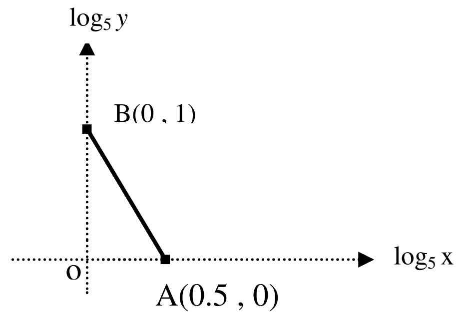
(b) $\int \frac{x^2}{(8 - x^3)^{\frac{1}{2}}}\, dx \quad = \frac{-2(8 - x^3)^{\frac{1}{2}}}{3} + C$ ★

$\int f''(x) = f(x) = \frac{-3}{2} \cdot \int \frac{x^2}{(8 - x^3)^{\frac{1}{2}}}\, dx$

$= \frac{-3}{2} \cdot \frac{-2(8 - x^3)^{\frac{1}{2}}}{3}$

$= (8 - x^3)^{\frac{1}{2}}$

11) The graph illustrates the law $y = kx^n$.
If the straight line passes through
A(0.5 , 0) and B(0 , 1), find the values of k and n.



Sol^n^

The gradient of the line is the value of n

$m_{AB} = n$

$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ A(0.5 , 0) B(0 , 1)

$= \frac{1-0}{0-0.5}$

$= \frac{1}{-0.5}$

$= \frac{1}{-1/2}$

$= 1 \cdot \frac{-2}{1}$

$= -2$

$n = -2$ ★

From the graph

$\log_5 x$	0	0.5
$\log_5 y$	1	0

$\log_5 x = 0$
$\log_5 5^0 = 0$
$x = 5^0 = 1$
$x = 1$

$\log_5 y = 1$
$\log_5 5^1 = 1$
$y = 5^1 = 5$
$y = 5$

$y = kx^n$
$5 = k \cdot 1^{-2}$
$1^{-2} = 1$
$5 = k$
$k = 5$ ★

$y = kx^n$
$y = 5x^{-2}$ ★

Alternatively,
$y = \frac{5}{x^2}$ ★