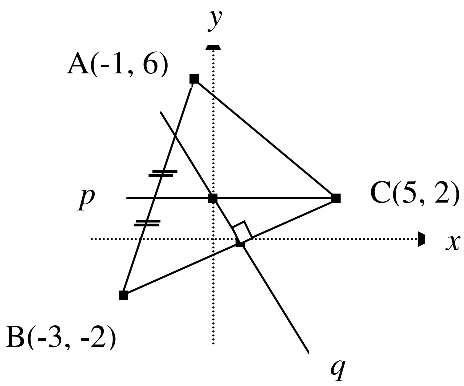
MATHEMATICS HIGHER

Units 1, 2 and 3

2002
Paper II

Questions
&
Worked Solutions

1) Triangle ABC has vertices A(-1, 6), B(-3, -2) and C(5, 2).



Find

(a) the equation of the line p, the median from C of triangle ABC.

(b) the equation of the line q, the perpendicular bisector of BC.

(c) the co-ordinates of the point of intersection of the lines p and q.

Sol[n]

(a) The line p, the median from C, goes through the point C and the mid-point of AB.
Find the mid-point of AB.

A(-1, 6)
B(-3, -2)

Let P be the mid-point of AB.

$$P\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{-1 + -3}{2}, \frac{6 + -2}{2}\right)$$

$$= \left(\frac{-4}{2}, \frac{4}{2}\right)$$

$$P = (-2, 2)$$

Find the gradient of CP

C(5, 2)
P(-2, 2)

$$m_{CP} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 2}{-2 - 5} = \frac{0}{-7} = 0$$

Equation of CP i.e the line p

C(5, 2) $m_{CP} = 0$

$y - b = m(x - a)$

$y - 2 = 0(x - 5)$

$y - 2 = 0$

$y = 2$

(b) The perpendicular bisector of BC, q — bisects BC; meets BC at right angles i.e. $m_q \bullet m_{BC} = -1$

Find the mid-point of BC

B(-3, -2)
C (5, 2)

Let Q be the mid-point of BC.

$$Q\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{-3 + 5}{2}, \frac{-2 + 2}{2}\right)$$

$$= \left(\frac{2}{2}, \frac{0}{2}\right)$$

$Q = (1, 0)$

Find the gradient of BC

B(-3, -2)
C (5, 2)

$$m_{BC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2 - (-2)}{5 - (-3)}$$

$$= \frac{2 + 2}{5 + 3}$$

$$= \frac{4}{8}$$

$$= \frac{1}{2}$$

Gradient of q,
the perpendicular bisector of BC = -2
i.e. $m_q = -2$

For perpendicular lines $m_1 \times m_2 = -1$ BC and q are perpendicular, $m_{BC} \times m_q = -1$ $\frac{1}{2} \times -2 = -1$ The gradient of q is -1

Equation of q, the perpendicular bisector of BC

C(1, 0) $m_q = -2$

$y - b = m(x - a)$

$y - 0 = -2(x - 1)$

$y = -2x + 2$

(c) Find the point of intersection of the lines p and q.

Find the point of intersection of $y = 2$

$y = -2x + 2$

$2 = -2x + 2$

$2x = 0$

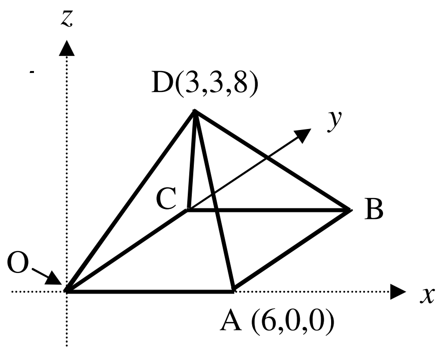
$x = 0$

The point of intersection of the lines p and q is (0, 2)

2) The diagram shows a square-based pyramid of height 8 units.

Square OABC has a side length of 6 units. The co-ordinates of A and D are (6, 0, 0) and (3, 3, 8).

C lies on the *y*-axis.



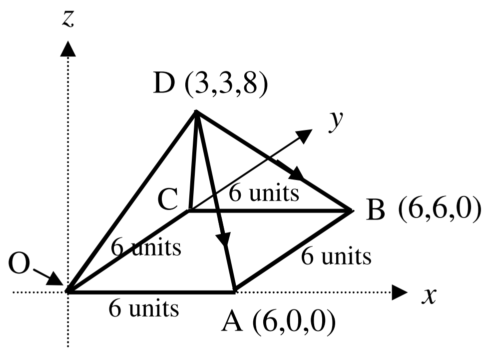
(a) Write down the co-ordinates of B.

(b) Determine the components of $\overrightarrow{DA}$ and $\overrightarrow{DB}$.

(c) Calculate the size of angle ADB.

Sol^n

(a) B is 6 units on the *x*-axis, 6 units on the *y*-axis and 0 units on the *z*-axis.
B is the point (6, 6, 0).



(b) $\overrightarrow{DA} = a - d$

$$= \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 6-3 \\ 0-3 \\ 0-8 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -3 \\ -8 \end{pmatrix}$$

$\overrightarrow{DB} = b - d$

$$= \begin{pmatrix} 6 \\ 6 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 6-3 \\ 6-3 \\ 0-8 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 3 \\ -8 \end{pmatrix}$$

From the above:

$\overrightarrow{DA} \cdot \overrightarrow{DB} = (3 \times 3) + (-3 \times 3) + (-8 \times -8)$
$= 9 + (-9) + 64$
$= 64$

$|\overrightarrow{DA}| = \sqrt{3^2 + (-3)^2 + (-8)^2} = \sqrt{9 + 9 + 64} = \sqrt{82}$

$|\overrightarrow{DB}| = \sqrt{3^2 + (3)^2 + (-8)^2} = \sqrt{9 + 9 + 64} = \sqrt{82}$

(c)

$$\cos \angle ADB = \frac{\overrightarrow{DA} \cdot \overrightarrow{DB}}{|\overrightarrow{DA}|\,|\overrightarrow{DB}|}$$

$$= \frac{(3 \times 3) + (-3 \times 3) + (-8 \times -8)}{\sqrt{(3^2 + (-3)^2 + (-8)^2} \times \sqrt{(3^2 + (3)^2 + (-8)^2}}$$

$$= \frac{9 + (-9) + 64}{\sqrt{9 + 9 + 64} \times \sqrt{9 + 9 + 64}}$$

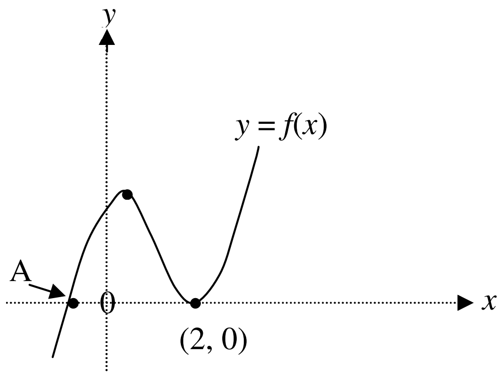
$$= \frac{64}{\sqrt{82} \times \sqrt{82}}$$

$$\cos \angle ADB = \frac{64}{82} = 0.780487804\ldots$$

$\angle ADB = 38.69473992\ldots$

$\angle ADB = 38.7^0$ (1 decimal place)

3) The diagram shows part of the graph of the curve with equation $y = 2x^3 - 7x^2 + 4x + 4$.



(a) Write down the x-coordinate of the maximum turning point.

(b) Factorise $2x^3 - 7x^2 + 4x + 4$.

(c) State the coordinates of the point A and hence find the values of x for which $2x^3 - 7x^2 + 4x + 4 < 0$.

Sol^n

(a) For stationary points $\frac{dy}{dx} = 0$

Differentiate the curve and set equal to zero

$y = 2x^3 - 7x^2 + 4x + 4$

$\frac{dy}{dx} = 6x^2 - 14x + 4$

Set $\frac{dy}{dx} = 0$

$6x^2 - 14x + 4 = 0$

$2(3x^2 - 7x + 2) = 0$

$2(3x - 1)(x - 2) = 0$

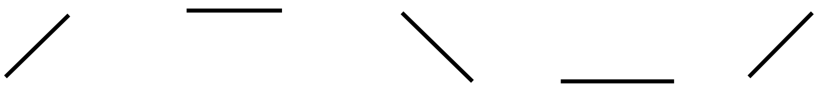
$(3x - 1) = 0$ or $(x - 2) = 0$

$x = \frac{1}{3}$ or $x = 2$

$\frac{dy}{dx} = 0$ when $x = \frac{1}{3}$ or $x = 2$ i.e. stationary points occur at $x = \frac{1}{3}$ or $x = 2$

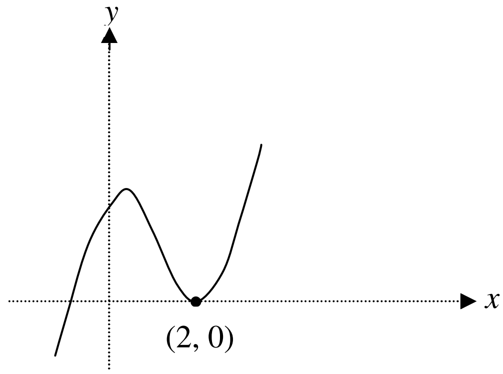
Nature of stationary points

x	0	$\frac{1}{3}$	1	2	3
$\frac{dy}{dx} = 2(3x - 1)(x - 2)$	$2(0 - 1)(0 - 2)$ $= 2(-1)(-2)$ $= 4$ positive	$2(1 - 1)(\frac{1}{3} - 2)$ $= 2(0)(\frac{-5}{3})$ $= 0$ zero	$2(3 - 1)(1 - 2)$ $= 2(2)(-1)$ $= -4$ negative	$2(6 - 1)(2 - 2)$ $= 2(5)(0)$ $= 0$ zero	$2(9 - 1)(3 - 2)$ $= 2(8)(1)$ $= 16$ positive



At $x = \frac{1}{3}$ there is a maximum turning point. The x-coordinate of the maximum turning point is $\frac{1}{3}$

(b) Factorise $2x^3 - 7x^2 + 4x + 4$.



From the diagram $\underline{x = 2}$ is one solution.

$(x - 2)$ is a factor of $2x^3 - 7x^2 + 4x + 4$.

Using the factor theorem

2	2	-7	4	4
		4	-6	-4
	2	-3	-2	0

$y = (x - 2)(2x^2 - 3x - 2)$

$2x^2 - 3x - 2 = (2x + 1)(x - 2)$

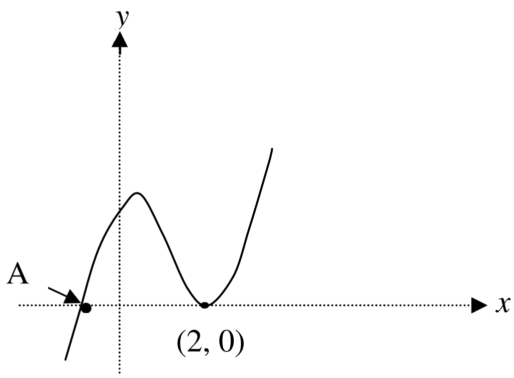
$y = (x - 2)(2x + 1)(x - 2)$

$y = (x - 2)^2(2x + 1)$

$y = 2x^3 - 7x^2 + 4x + 4 = (x - 2)^2(2x + 1)$

(c) Set $y = 0$ $(x - 2) = 0$ or $(2x + 1) = 0$

$x = 2$ or $x = \frac{-1}{2}$



The points (2, 0) and $\left(\frac{-1}{2}, 0\right)$

are on the curve $y = 2x^3 - 7x^2 + 4x + 4$.

A is the point $\left(\frac{-1}{2}, 0\right)$

$y = 2x^3 - 7x^2 + 4x + 4$

$2x^3 - 7x^2 + 4x + 4 < 0$ when $x < \frac{-1}{2}$

(4) A man decides to plant a number of fast-growing trees as a boundary between his property and the property of his next door neighbour. He has been warned, however, by the local garden centre that, during any year the trees are expected to increase in height by 0.5 metres. In response to this warning he decides to trim 20% off the height of the trees at the start of any year.

(a) If he adopts the "20% pruning policy", to what height will he expect the trees to grow in the long run?

(b) His neighbour is concerned that the trees are growing at an alarming rate and wants assurances that the trees will grow no taller than 2 metres. What is the minimum percentage that the trees will need to be trimmed each year so as to meet this condition?

Sol^n

(a) At the start of any year, after pruning 20% off the trees, the man is left with **80%** of the height of the trees.

$80\% = 0.80 = 0.8$

Also, during the previous year, the trees have grown **0. 5** metres.

Recurrence relation $u_{n+1} = 0.8u_n + 0.5$

$$L = \frac{0.5}{1 - 0.8} = \frac{0.5}{0.2} = \frac{5}{2} = 2.5$$

$u_{n+1} = au_n + b$

$a = 0.8 \qquad b = 0.5$

$-1 < 0.8 < 1$, $\therefore$ a limit exists

$$L = \frac{b}{1 - a}$$

If the man adopts a "20% pruning policy" he will expect the trees, in the long run, to grow to a height of **2.5** metres.

(b)

The neighbour wants the trees to grow to a height of no more than 2 metres. Set L = 2
The trees are still expected, during any year, to increase in height by 0.5 metres. b = 0.5
Require the minimum percentage that the trees need to be trimmed each year for a Limit of 2 metres. Find the value of a.

$$L = \frac{b}{1 - a}$$

$L = 2, \ b = 0.5$

$$2 = \frac{0.5}{1 - a}$$

(x by 1 – a both sides)

$2(1 - a) = 0.5$

(divide by 2 both sides)

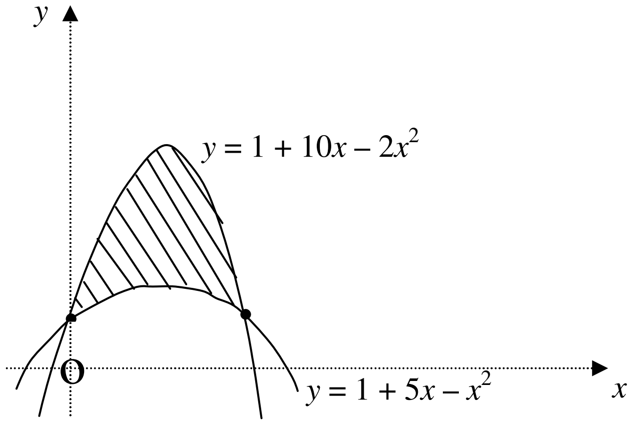
$1 - a = 0.25$

$1 - 0.25 = a$

$a = 0.75$

For the trees to grow no taller than 2 metres, the recurrence relation is $u_{n+1} = 0.75u_n + 0.5$. The man is left with 75% of the height of the trees. To not exceed 2 metres, the minimum % for trimming each year is **25%**.

5) Calculate the shaded area enclosed between the parabolas with equations $y = 1 + 10x - 2x^2$ and $y = 1 + 5x - x^2$.



<u>Sol^n</u>

Find the x-coordinates of the pints of intersection of $y = 1 + 10x - 2x^2$ and $y = 1 + 5x - x^2$.

<u>Set equal</u>

i.e. $1 + 10x - 2x^2 = 1 + 5x - x^2$

$10x - 2x^2 = 5x - x^2$

$0 = x^2 - 5x$

$x^2 - 5x = 0$

$x(x - 5) = 0$

$\underline{\underline{x = 0}}$ or $x - 5 = 0$

$\underline{\underline{x = 5}}$

$$\int_0^5 1 + 10x - 2x^2 - \left(1 + 5x - x^2\right) \, dx$$

$$= \int_0^5 1 + 10x - 2x^2 - 1 - 5x + x^2 \, dx$$

$$= \int_0^5 5x - x^2 \, dx$$

$$= \left(\frac{5x^2}{2} - \frac{x^3}{3}\right)_0^5 \qquad = \left(\frac{5(5)^2}{2} - \frac{5^3}{3}\right) - \left(\frac{5(0)^2}{2} - \frac{0^3}{3}\right)$$

$$= \left(\frac{125}{2} - \frac{125}{3}\right) - \left(\frac{0}{2} - \frac{0}{3}\right)$$

$$= \frac{375}{6} - \frac{250}{6}$$

$$= \frac{125}{6}$$

$$= \underline{\underline{20\tfrac{5}{6} \text{ units}^2}}$$

6) Find the equation of the tangent to the curve $y = 2\sin\left(x - \frac{\pi}{6}\right)$ at the point where $x = \frac{\pi}{3}$.

<u>Sol</u>^{n}

$x = \frac{\pi}{3}$

$$y = 2\sin\left(x - \frac{\pi}{6}\right)$$

$$= 2\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

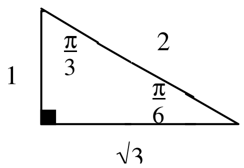
$$= 2\sin\left(\frac{2\pi}{6} - \frac{\pi}{6}\right)$$

$$= 2\sin\left(\frac{\pi}{6}\right)$$

$$= 2 \cdot \frac{1}{2}$$

$$= 1$$

$\sin\frac{\pi}{6} = \frac{1}{2} = 0.5$



The point $\left(\frac{\pi}{3}, 1\right)$ lies on both the curve and the tangent to the curve.

<u>Find the gradient of the tangent to the curve at any point.</u>

i.e. <u>differentiate</u> $y = 2\sin\left(x - \frac{\pi}{6}\right)$

$$\frac{dy}{dx} = 2\cos\left(x - \frac{\pi}{6}\right)$$

$x = \frac{\pi}{3}$

$$= 2\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

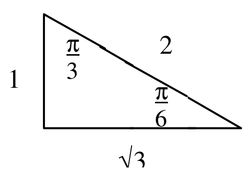
$$= 2\cos\left(\frac{2\pi}{6} - \frac{\pi}{6}\right)$$

$$= 2\cos\left(\frac{\pi}{6}\right)$$

$$= 2 \cdot \frac{\sqrt{3}}{2}$$

$$= \sqrt{3}$$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$



At the point $\left(\frac{\pi}{3}, 1\right)$ the gradient of the tangent to the curve is $\sqrt{3}$

Find the equation of the tangent.

point $\left(\frac{\pi}{3}, 1 \right)$

$m_{tan} = \sqrt{3}$

$y - b = m(x - a)$

$y - 1 = \sqrt{3}\left(x - \frac{\pi}{3} \right)$

$y - 1 = \sqrt{3}x - \sqrt{3} \cdot \frac{\pi}{3}$

(x by 3 both sides)

$3y - 3 = 3 \cdot \sqrt{3}x - \sqrt{3}\pi$

$3y = 3 \cdot \sqrt{3}x - \sqrt{3}\pi + 3$

(divide by 3 both sides)

$y = \sqrt{3}x - \frac{\sqrt{3}\pi}{3} + 1$

$y = \sqrt{3}x + 1 - \frac{\pi}{\sqrt{3}}$

7) Find the x-coordinate of the point where the graph of the curve with equation $y = \log_3(x-2) + 1$ intersects the x-axis.

<u>Sol^n</u>

The graph of $y = \log_3(x-2) + 1$ intersects the x-axis when $y = 0$.

$\log_3(x-2) + 1 = 0$
$\log_3(x-2) = -1$

$(x-2) = 3^{-1}$

$(x-2) = \frac{1}{3}$

$x = \frac{1}{3} + 2$

$x = 2\frac{1}{3}$ or $x = \frac{7}{3}$

From above, the graph of $y = \log_3(x-2)$ intersects the x-axis at the point $\left(\frac{7}{3}, 0\right)$

8) A point moves in a straight line such that its acceleration a is given by $a = 2(4-t)^{\frac{1}{2}}$, $0 \le t \le 4$. If it starts at rest, find an expression for the velocity v where $a = \frac{dv}{dt}$.

Sol^n^

$$v = \int 2(4-t)^{\frac{1}{2}}\, dt$$

$$= 2\int (4-t)^{\frac{1}{2}}\, dt$$

$$= 2 \cdot \frac{(4-t)^{\frac{3}{2}}}{\frac{3}{2}} \cdot (-1) + C$$

$$= 2 \cdot \frac{2}{3} \cdot (-1) \cdot (4-t)^{\frac{3}{2}} + C$$

$$v = \frac{-4(4-t)^{\frac{3}{2}}}{3} + C$$

Find the value of C

$$v = \frac{-4(4-t)^{\frac{3}{2}}}{3} + C$$

When $t = 0$, $v = 0$

$$0 = \frac{-4(4-0)^{\frac{3}{2}}}{3} + C$$

$$0 = \frac{-4(4)^{\frac{3}{2}}}{3} + C$$

Change side, change sign

$$\frac{4(\sqrt{4})^3}{3} = C$$

$$C = \frac{(4)(2)^3}{3}$$

$$C = \frac{(4)(8)}{3}$$

$$C = \frac{32}{3}$$

$$v = \frac{-4(4-t)^{\frac{3}{2}}}{3} + C$$

$$v = \frac{-4(4-t)^{\frac{3}{2}}}{3} + \frac{32}{3}$$

$$v = \frac{-4}{3}\left((4-t)^{\frac{3}{2}} - 8\right)$$

9) Show that the equation $(1 - 2k)x^2 - 5kx - 2k = 0$ has real roots for all integer values of k.

<u>Sol</u>^{n}

$(1 - 2k)x^2 - 5kx - 2k = 0$ has real roots given $b^2 - 4ac \geq 0$.

$a = 1 - 2k$
$b = -5k$
$c = -2k$

$$\begin{aligned} b^2 - 4ac &= (-5k)^2 - 4(1 - 2k)(-2k) \\ &= 25k^2 - (4 - 8k)(-2k) \\ &= 25k^2 + 2k(4 - 8k) \\ &= 25k^2 + 8k - 16k^2 \\ &= 9k^2 + 8k \end{aligned}$$

$9k^2 + 8k \geq 0$ for all integer values of k.

$\therefore b^2 - 4ac \geq 0$ for all integer values of k.

$\therefore (1 - 2k)x^2 - 5kx - 2k = 0$ has real roots for all integer values of k.

<u>$9k^2 + 8k \geq 0$ for all integer values of k</u>

$y = 9k^2 + 8k$ is a parabola

<u>Find the zeros of the function (i.e. set $y = 0$)</u>

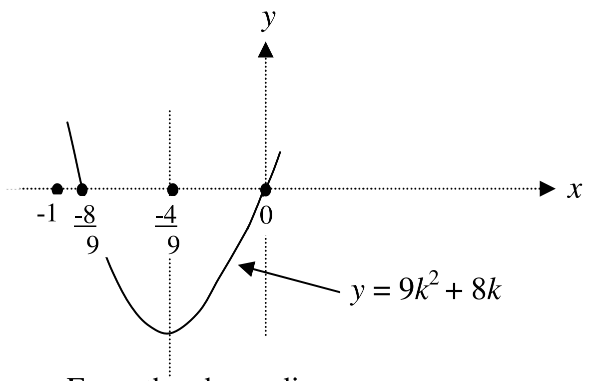
$9k^2 + 8k = 0$

$k(9k + 8) = 0$

$k = 0$ or $9k + 8 = 0$

$k = \frac{-8}{9}$

$(0,0)$ and $\left(\frac{-8}{9}, 0\right)$ lie on the curve $y = 9k^2 + 8k$.



From the above diagram

$9k^2 + 8k < 0$ for $\frac{-8}{9} < k < 0$

k is <u>not</u> an integer when $\frac{-8}{9} < k < 0$

$\therefore$ $9k^2 + 8k < 0$ when k is <u>not</u> an integer.

$\therefore$ $9k^2 + 8k \geq 0$ for all integer values of k.

<u>Find the stationary value</u>

$\frac{dy}{dk} = 18k + 8$

<u>Set</u> $\frac{dy}{dk} = 0$

$18k + 8 = 0$

$k = \frac{-8}{18} = \frac{-4}{9}$

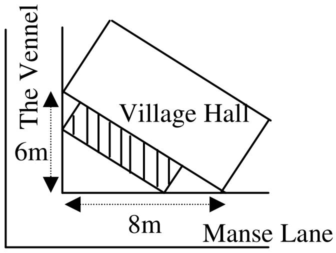
Stationary value when $k = \frac{-4}{9}$

<u>Nature of stationary point</u>

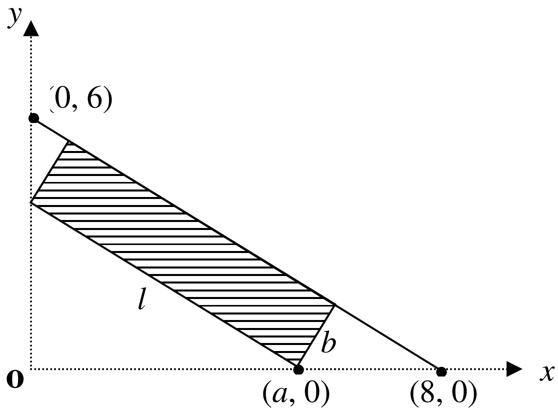
k	-1	$\frac{-4}{9}$	0
$\frac{dy}{dk} = 18k + 8$	$-18+8$ $= -10$ neg.	0 zero	$0+8$ $= 8$ pos.
	\	—	/

$y = 9k^2 + 8k$ has a <u>minimum turning point</u> when $k = \frac{-4}{9}$

10) The shaded rectangle on this map represents the planned extension to the village hall. It is hoped to provide the largest possible area for the extension.



The coordinate diagram represents the right angled triangle of ground behind the hall. The extension has length l metres and breadth b metres, as shown. One corner of the extension is at the point $(a, 0)$.



(a) (i) Show that $l = \frac{5}{4}a$.

(ii) Express b in terms of a and hence deduce that the area, A m^2, of the extension is given by $A = \frac{3}{4}a(8 - a)$.

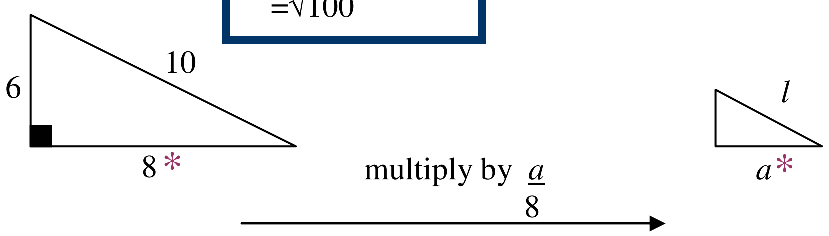
(b) Find the value of a which produces the largest area of the extension.

Sol^n

(a) (i) From the 2nd diagram above

Similar Triangles

By Pythagoras
$10 = \sqrt{6^2 + 8^2}$
$= \sqrt{36 + 64}$
$= \sqrt{100}$

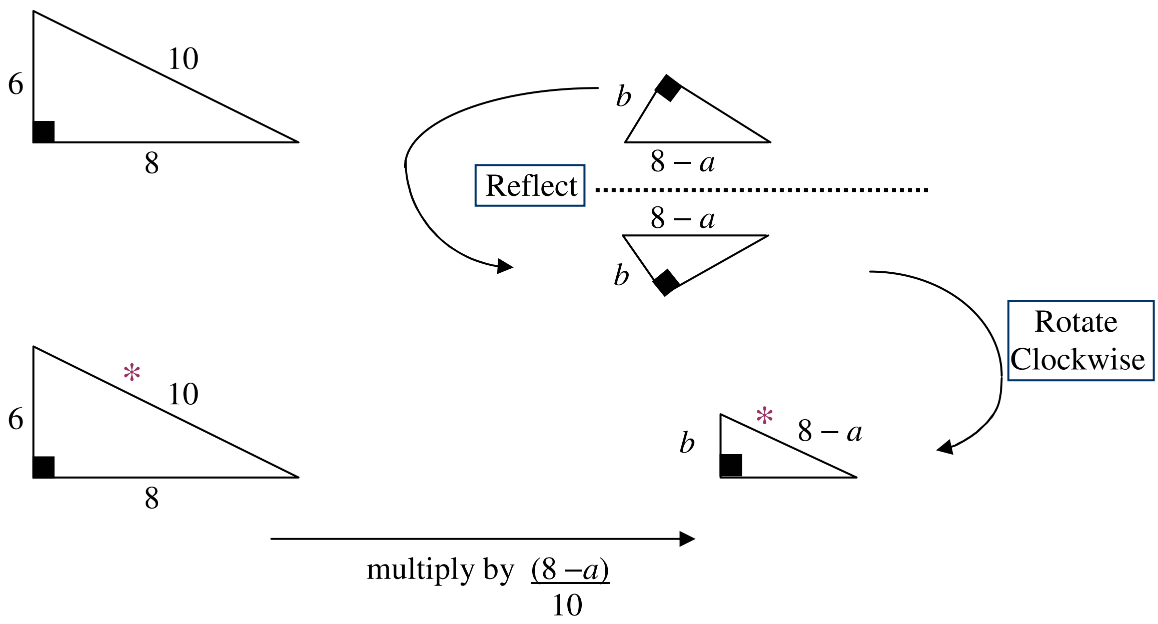


RSF (Reduction Scale Factor) $= \frac{\text{small}}{\text{big}} = \frac{a}{8}$

$l = 10 \bullet \frac{a}{8} = \frac{5}{4}a$

$l = \frac{5}{4}a$

(a) (ii) <u>From the 2nd diagram</u>
<u>Similar Triangles</u>



RSF (Reduction Scale Factor) $= \frac{\text{small}}{\text{big}} = \frac{(8-a)}{10}$

$b = 6 \bullet \frac{(8-a)}{10} = \frac{3}{5} \bullet (8-a)$

$b = \frac{3}{5}(8-a)$ ★

The area of the extension, A m^2, is given by the formula

$A = lb$

$= \frac{5}{4}a \bullet \frac{3}{5}(8-a)$

$= \frac{3}{4}a(8-a)$

$l = \frac{5}{4}a$

$b = \frac{3}{5}(8-a)$

$A = \frac{3}{4}a(8-a)$ ★

(b) Find the value of *a* which produces the largest area of the extension.

$$A = \frac{3}{4}a(8-a)$$
$$= \frac{24a}{4} - \frac{3a^2}{4}$$
$$= 6a - \frac{3a^2}{4}$$

$$A = 6a - \frac{3}{4}a^2$$

<u>Differentiate</u>

$$\frac{dA}{da} = 6 - \frac{6}{4}a$$
$$= 6 - \frac{3}{2}a$$

Set $\frac{dA}{da} = 0$

$$6 - \frac{3}{2}a = 0$$
(x by 2 both sides)
$$12 - 3a = 0$$
$$12 = 3a$$
$$3a = 12$$
$$a = 4$$

For $A = 6a - \frac{3}{4}a^2$, a stationary point occurs when $a = 4$.

<u>Check nature of the stationary point</u>

a	3	4	5
$\frac{dA}{da} = 6 - \frac{3}{2}a$	$6 - \frac{9}{2}$ $= 6 - 4.5$ $= 1.5$ positive	$6 - \frac{12}{2}$ $= 6 - 6$ $= 0$ zero	$6 - \frac{15}{2}$ $= 6 - 7.5$ $= -1.5$ negative
	/	—	\

$A = 6a - \frac{3}{2}a^2$ has a maximum turning point at $a = 4$

i.e. $a = 4$ is the value of *a* which produces the largest area of the extension.